Int. J. Heat Mass Transfer. Vol. 30, No. 7, pp. 1371–1380, 1987
Printed in Great Britain

0017–9310/87 $3.00+0.00

Anisotropic scattering in three-dimensional differential approximation for radiation heat transfer

DUANE W. CONDIFF
Reactor Analysis and Safety Division, Argonne National Laboratory, Argonne, IL 60439, U.S.A.

(*Received* 10 *July* 1986 *and in final form* 5 *November* 1986)

Abstract—The differential approximation is extended to account for anisotropic scattering in invariant three-dimensional form. A moment method using polyadic Legendre functions establishes that pressure cross-sections should take precedence over extinction cross-sections for treating radiation heat transfer in an absorbing, emitting, and scattering medium, and that use of these cross-sections accounts for the extent of preferred forward or backward scattering. The resulting generalized differential approximation is discussed in several simple example problems.

1. INTRODUCTION

A RELATIVELY simple yet general method of treating thermal radiation heat transfer within an absorbing, emitting and scattering medium with and without other modes of heat transfer is clearly needed and should have broad applicability. In particular, it is of significance to current problems involving nuclear reactor accidents under conditions of severe core meltdown and dispersal of high-temperature airborne debris within containment, wherein the removal of heat from clouds of molten and particulate aerosol by thermal radiation with combined absorption, emission, and scattering, is an important safety consideration.

In recent years the multi-dimensional aspects of such phenomena have been studied using various special methods. For example, the P–N approximation has been employed extensively by Mengüc and Viskanta [1] and by Ratzel and Howell [2]. The method of finite elements has been applied by Razzaque *et al.* [3, 4] and by Fernandes *et al.* [5], cf. also Fernandes and Francis [6], as well as Wu *et al.* [7]. Solutions to the radiative transport equation by discrete ordinates have been obtained by Fiveland [8].

These approaches are relatively tedious to apply in comparison to the differential approximation, which as a powerful useful generalization of the classical Rosseland diffusion approximation to arbitrary optical depths, for an absorbing and emitting medium is well known [9, 10]. This method was extended by Azad and Modest [11–14] to include linear anisotropic scattering and shown to agree well with the exact solution to the corresponding radiative transport equation for this case. With new interpretations the calculations and solutions reported for linear anisotropic scattering using the differential approximation should all apply to general anisotropic scattering phase functions, and also offer some confidence in the accuracy of the method when generalized in the absence of exact solutions.

Such a generalized formulation of the method is appropriate because most single scattering even by spherical particles is anisotropic with considerable skewing or preferences for the forward or backward directions relative to incident radiation. In this paper we present (Section 1), derive (Section 2), and exemplify (Section 3), a generalized differential approximation for isotropic media. The formulation, which includes anisotropic scattering, is presented and derived in coordinate-system invariant form suitable for application to three-dimensional problems. Properly applied it should prove useful for development and extension of computer codes as well as in analytic studies.

Section 1 exhibits the differential approximations with extinction cross-sections of isotropic scattering replaced by pressure cross-sections. The relationship to Mie scattering is described, illustrating that only the pressure cross-section possesses proper invariance to the inclusion of forward scattering. Section 2 includes a derivation of the formulation using polyadic Legendre functions while Section 2.1 describes its reduction to the linear anisotropic form and the one-dimensional radiative transport equation. Section 2.2 derives the jump boundary conditions using polyadic Legendre functions. Section 3 is concerned with simple solutions, including a discussion of when and how existing solutions for isotropic and linear anisotropic scattering can be adapted in the differential approximation to general anisotropic scattering. Then specific simple solutions are developed which incorporate general scattering with temperature stabilized by phase changes in plane, cylindrical, and spherical geometries (Sections 3.1–3.3). Section 4 describes the invariance of the basic radiative transport equation to a transformation involving forward scatter, as a generalization and clarification of the invariance principle for pressure cross-sections described in Section 1.

1.1. *Formulation*

A general three-dimensional derivation is presented in Section 2 which shows that the proper moment

NOMENCLATURE

a	absorption coefficient
b	equation (35)
c	constant
e_b	black body emissive power
I	modified Bessel functions of first kind
i	radiative directional intensity
$\mathbf{i}$	expansion coefficient functions for i (equation (9))
K	extinction coefficient $(a+s)$
$\hat{n}$	unit vector normal to surface
n	particle number density
$\mathbf{P}_l$	polyadic Legendre function, order l
P_l	scalar Legendre function, order l
$\mathbf{q}$	radiative heat flux
Q_{pr}	radiation pressure efficiency factor
Q_{abs}	absorption efficiency factor
Q_{ext}	extinction efficiency factor
$\mathbf{r}$	position vector
R	radius
s	scattering coefficient
S	source density
t	time
T	temperature.

Greek symbols

α	liquid volume fraction
β	equations (20), (21) and (24)
δ	mean value of cos (θ), Kronecker delta
ε	emissivity
η	scalar eigenvalues
θ	scattering angle
λ	latent heat
μ	equations (19) and (22)
ρ	mass density
σ	Stefan–Boltzmann constant
Φ	scattering phase function
ψ	$\Phi/4\pi$, also equation (47) or (58)
$\hat{\omega}$	unit directional vector
$d\omega$	differential solid angle.

Subscripts

b	black body
λ	spectral (wavelength specific)
i	incident
l	integer
n	outward normal to surface
$-n$	inward normal to surface
w	wall or boundary value
R	spherical or cylindrical interface
m	melting point.

equation to be associated with the differential approximation for the radiant spectral heat flux vector $\mathbf{q}_\lambda$ in an isotropic medium has the form

$$\nabla(\nabla\cdot\mathbf{q}_\lambda)-3a_\lambda(K_\lambda-s_\lambda\delta_\lambda)\mathbf{q}_\lambda = 4a_\lambda\nabla e_{b\lambda} \tag{1}$$

where $e_{b\lambda}$ is the spectral black body emission, a_λ the absorption coefficient, s_λ the total scattering coefficient so that $K_\lambda = a_\lambda+s_\lambda$ is the local extinction coefficient. For an optically thick medium this result reduces to a generalized form of the classical Rosseland diffusion approximation

$$\mathbf{q}_\lambda = \frac{-4\nabla e_{b\lambda}(T,\lambda)}{3(K_\lambda-s_\lambda\delta_\lambda)} \tag{2}$$

where gradient operators are in physical, as opposed to optical, coordinates. The apparently new term to heat transfer considerations, applicable quite unrestrictively as shown below, is $s_\lambda\delta_\lambda$, with δ_λ defined by

$$\delta_\lambda = \overline{\cos\theta} \quad (-1<\delta_\lambda<1) \tag{3}$$

as the average of the scattering angle θ between incident and scattered directions computed using the scattering phase function, cf. equation (3.1). For scattering and absorbing spheres of radius R, and number density n, we have from Van de Hulst [15]

$$K_\lambda-s_\lambda\delta_\lambda = \pi R^2 nQ_{pr} \tag{4}$$

where

$$Q_{pr} = Q_{ext}-Q_{sca}\overline{\cos\theta} = Q_{abs}+Q_{sca}(1-\overline{\cos\theta}) \tag{5}$$

is the pressure efficiency factor expressed in terms of extinction and scattering, or absorption and scattering efficiency factors. Thus it is the pressure cross-section which properly appears in the differential and diffusion forms of equations (1) and (2).

Figure 1 schematically depicts the range of values permitted. For isotropic scattering, or when there is equal forward and backward scattering, $\delta_\lambda = 0$, hence, the extinction and pressure cross-sections become equal and the former is applicable as is commonly assumed. Small dielectric particles, for example, have $\delta_\lambda < 0$ corresponding to a preponderance of back scatter, which diminishes the effective radiant thermal conductivity in equation (2) or equation (1). Large dielectric spheres, in contrast, have predominantly forward scatter so $\delta_\lambda > 0$, which enhances the effective conductivity as expected. The importance of these distinctions increases as scattering albedo $\Omega_\lambda = s_\lambda/K_\lambda$ approaches unity, and clearly becomes insignificant for Ω_λ near zero (case b) where absorption dominates over all scattering. Even when Ω_λ is not near zero, Fig. 1 shows that the differential approximation described by equation (1) embodies the intuitive expectations that absorption would dominate if forward scatter were so predominant that Ω_λ approaches unity.

Because of the relationship, equation (5), to pres-

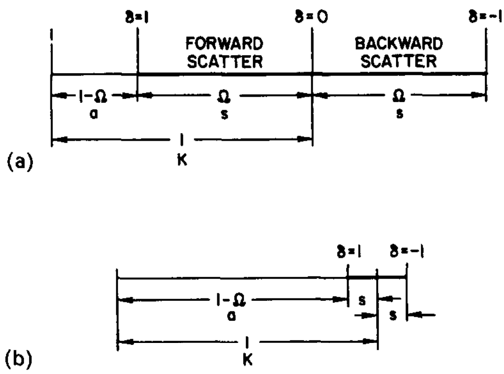

FIG. 1. (a) Large scattering albedo, Ω. (b) Small scattering albedo.

Table 1. Comparison of pressure and extinction cross-sections for water droplets in air at $\lambda = 1.5\ \mu m$, $n = 1.32–0.006i$

Radius (μm)	Q_{abs}	Q_{ext}	Q_{pr}
1	0.100	2.83	0.535
2	0.203	3.22	0.720
5	0.435	2.08	0.692
10	0.620	2.06	0.757
20	0.825	2.11	0.892
50	0.951	2.05	0.984
100	0.952	2.04	0.981
200	0.944	2.02	0.973
500	0.989	2.01	0.968

sure cross-sections, as well as the simplicity of the integrations for explicit scattering phase functions Φ_λ, some information is readily available to determine δ_λ computationally and analytically, via integration

$$\overline{\cos\theta} = \frac{1}{2}\int_0^\pi \Phi_\lambda(\theta)\sin\theta\cos\theta\,d\theta. \qquad (3.1)$$

We note limiting values of $-4/9$, and $-2/5-x^2/15$ with $x = 2\pi R/\lambda$, easily determined respectively for diffuse spheres (Lambert reflectors), and small highly reflective conducting spheres. Early tabulated values of $\overline{\cos\theta}$ were computed by Debeye, cf. ref. [15], p. 226. Numerous values of Q_{pr} or $\overline{\cos\theta}$ are also reported therein at assorted refractive indices, e.g. pp. 276, 280, 161, and Hottel and Sarofim [16] (p. 389) have provided a convenient plot of $\overline{\cos\theta}$ vs real refractive index for large transparent spheres. For intermediate sizes of non-absorbing spheres ($\Omega \to 1$), ref. [15] indicates that δ_λ changes sign from backward to forward scattering at a size parameter $x = 1.38$, cf. also ref. [16], p. 406, for a corresponding plot of $\overline{\cos\theta}$ over a full range of x. General computations of Φ_λ, and hence δ_λ, for arbitrary absorbing and scattering spheres with non-asymptotic values of complex refractive index require application of the Mie scattering solution, cf. refs. [15, 16].

Table 1 depicts sample values of the absorption, extinction, and pressure efficiency factors for water droplets from 1 to 500 μm at a wavelength of 1.5 μm, calculated from the Mie solution. This demonstrates that extinction coefficients tend to be factors of 2–3 larger than the pressure coefficients. Thus, the use of Q_{pr} is quantitatively very significant to heat transfer calculations.

A clarification on the relationship of the Mie solution to present formulations for heat transfer considerations is appropriate. For determination of any of the scattering cross-sections, the Mie solution contains contributions to scattering associated with diffraction which are inseparable from refraction–reflection contributions except in the asymptotic limit of particles sufficiently large that geometric ray optics is applicable. In this limit the separable diffraction component becomes pure forward scatter, i.e. contributes a Dirac delta function $\delta(\cos\theta-1)$ component to the scattering phase function, as well as defines a contribution to the total scattering coefficient s_λ. It appears customary to omit the diffraction part of s_λ in applying the extinction coefficient $K_\lambda = a_\lambda + s_\lambda$ to heat transfer considerations because undeflected forward scatter is thermally equivalent to no scatter. This, however, presents an ambiguity in applying the Mie solution for intermediate size particles, i.e. removed from the ray optics limit, in determining applicable values of s_λ, K_λ. In such circumstances the diffraction contribution cannot be separated and must not be omitted since it contributes to deflected forward scatter; but then the limit of ray optics values for s_λ and K_λ is not approached.

This ambiguity is eliminated when the extinction coefficient K_λ is properly replaced by $K_\lambda - s_\lambda\delta_\lambda$ in heat transfer considerations as defined in equation (1) or equation (2), as this coefficient automatically screens out contributions from undeflected forward scatter. To demonstrate this, consider the ray optics limit with and without a diffraction term. Consider non-absorbing spheres with phase function $\Phi_\lambda(\theta)$ which does not include diffraction, and associated scattering coefficient s_λ. The value of δ_λ is then determined by equations (3) and (3.1). If diffraction is now included, s_λ must be replaced by $2s_\lambda$ and Φ_λ by $\Phi_\lambda/2 + \delta(\cos\theta - 1)$. Applying these to equations (3) and (3.1) the value of $s_\lambda(1-\delta_\lambda)$ thereby becomes

$$2s_\lambda(1-\delta_\lambda/2-1/2) \equiv s_\lambda(1-\delta_\lambda).$$

This invariance of $K_\lambda - s_\lambda\delta_\lambda$ to the inclusion of forward diffraction scatter confirms the validity of the choice $K_\lambda - s_\lambda\delta_\lambda$ over the total extinction coefficient K_λ for use in heat transfer considerations involving the differential approximation.

2. DERIVATION OF DIFFERENTIAL APPROXIMATION WITH SCATTERING

Our starting point is the field equation form of the radiative transfer equation for spectral intensity $i(\mathbf{r}, \hat{\omega})$

$$\hat{\omega}\cdot\nabla i = -Ki + ai_b + s\int i(\mathbf{r},\hat{\omega}_i)\psi(\hat{\omega}_i\cdot\hat{\omega})\,d\omega_i. \quad (6)$$

Here $\mathbf{r}$ is a physical location vector, $\hat{\omega}$ is a unit vector in the direction of intensity i, $\hat{\omega}_i$ is the same for incident rays scattered in direction $\hat{\omega}$, and $d\omega_i$ designates solid angle integration; $i_b(\mathbf{r}) \equiv e_b(\mathbf{r})/\pi$ denotes Planck black body emission intensity corresponding to local temperature $T(\mathbf{r})$. The extinction, absorption, and scattering coefficients K, a, s are as defined above, with λ subscripts suppressed. The principal assumption of equation (6) is that the medium is isotropic, so that the scattering phase function $\psi \equiv \Phi/4\pi$ is an isotropic scalar function of $\hat{\omega}_i$ and $\hat{\omega}$, i.e. is invariant to coordinate system rotations, which implies it depends only upon the scattering angle or $\cos\theta = \hat{\omega}_i\cdot\hat{\omega}$. (This isotropic medium assumption is, of course, quite apart from an isotropic scattering condition according to which ψ is constant or $\Phi = 1$, so that the amount of scattering of an incident ray would be uniform in all directions. That assumption is relaxed here.)

Our method is to employ an expansion in irreducible tensors, i.e. polyadic Legendre functions of the unit vector $\hat{\omega}$. As such, it may be considered an invariant form of the moment method or method of spherical harmonics, but has the distinct advantage of not requiring any choice of coordinate system which is especially useful for three-dimensional considerations.

Polyadic Legendre functions of a unit vector $\hat{\omega}$ were first defined as such by Brenner [17]. Their analogy to scalar Legendre functions $P_l(Z)$ may be seen in the following:

$$\begin{aligned}
&P_0(Z) = 1 && \mathbf{P}_0(\hat{\omega}) = 1\\
&P_1(Z) = Z && \mathbf{P}_1(\hat{\omega}) = \hat{\omega}\\
&P_2(Z) = \frac{3}{2}Z^2 - \frac{1}{2} && \mathbf{P}_2(\hat{\omega}) = \frac{3}{2}\hat{\omega}\hat{\omega} - \frac{1}{2}\mathbf{U}\\
&P_3(Z) = \frac{5}{2}Z^3 - \frac{3}{2}Z && \mathbf{P}_3(\hat{\omega}) = \frac{5}{2}\hat{\omega}\hat{\omega}\hat{\omega} - \frac{3}{2}\langle\hat{\omega}\mathbf{U}\rangle_S\\
&\quad\vdots && \quad\vdots
\end{aligned} \quad (7)$$

where $\mathbf{U}$ is the second-order unit dyadic and $\langle\ \rangle_S$ denotes symmetrization with respect to all pairs of tensorial indices. Thus, $\mathbf{P}_l(\hat{\omega})$ is an irreducible tensor or polyadic of order l, i.e. both symmetric and traceless with respect to any pair of tensorial indices. Consequently, these satisfy the scalar relationships

$$(\hat{\omega}\hat{\omega}\ldots\hat{\omega})_l(\cdot)^l\mathbf{P}_l(\hat{\omega}_i) = \frac{2^l(l!)^2}{(2l)!}\mathbf{P}_l(\hat{\omega})(\cdot)^l\mathbf{P}_l(\hat{\omega}_i) = P_l(\hat{\omega}\cdot\hat{\omega}_i) \quad (8)$$

where $(\cdot)^l$ denotes an lth order tensorial contraction or dot product to scalar result. A scalar function such as intensity $i(\mathbf{r},\hat{\omega})$ defined over the unit sphere range of $\hat{\omega}$ may be represented by the expansion

$$i(\mathbf{r},\hat{\omega}) = i_0(\mathbf{r}) + \sum_{l=1} \mathbf{i}_l(\mathbf{r})(\cdot)^l\mathbf{P}_l(\hat{\omega}) \quad (9)$$

with polyadic expansion coefficients $\mathbf{i}_l(\mathbf{r})$, irreducible tensors determined by orthogonality and normalization of $\mathbf{P}_l(\hat{\omega})$ as

$$\mathbf{i}_l(\mathbf{r}) = \frac{(2l+1)(l!)^2 2^l}{4\pi(2l)!}\int i(\mathbf{r},\hat{\omega})\mathbf{P}_l(\hat{\omega})\,d\omega. \quad (10)$$

In particular, we recover for $l = 1, 0, 2$

$$\mathbf{i}_1(\mathbf{r}) = \frac{3}{4\pi}\int i\hat{\omega}\,d\omega = \frac{3}{4\pi}\mathbf{q} \quad (11)$$

with $\mathbf{q}$ the radiant heat flux vector; also that $i_0(\mathbf{r})/3$ represents the equilibrium portion of the radiation pressure, and that $\mathbf{i}_2(\mathbf{r})$ represents the non-equilibrium traceless or shear portion of the radiation pressure tensor.

Additional characteristics of such expansions for isotropic tensor functions of $\hat{\omega}$ with applicability to kinetic theory of polyatomic gases with anisotropic collision cross-sections and their effects on gas transport coefficients were described previously [18].

An important property of the scattering integral operator contained in equation (6) is that the $\mathbf{P}_l(\hat{\omega})$ are tensorial eigenfunctions of it with scalar eigenvalues, η_l, namely

$$\int \psi(\hat{\omega}_i\cdot\hat{\omega})\mathbf{P}_l(\hat{\omega}_i)\,d\omega_i = \eta_l\mathbf{P}_l(\hat{\omega}). \quad (12)$$

This follows from the fact that the integral result must at once be an isotropic tensor function of $\hat{\omega}$, and be irreducible, i.e. traceless and symmetric in any tensor index pair. These conditions uniquely qualify $\mathbf{P}_l(\hat{\omega})$ to within the constant η_l. Such scalar eigenvalues may be found by multiplying both sides by $(\cdot)^l\mathbf{P}_l(\hat{\omega})\cdot(2^l(l!)^2/(2l)!)$ to obtain using equation (8) and the fact that $P_l(1) = 1$

$$\eta_l = \int \psi(\cos\theta)P_l(\cos\theta)\,d\omega_i = \overline{P_l(\cos\theta)}. \quad (13)$$

Thus η_l is the mean value of the lth scalar Legendre function of $\cos\theta$, where θ is the scattering angle. In particular we have

$$\eta_0 = 1, \quad \eta_1 = \overline{\cos\theta}, \quad \eta_2 = \frac{3}{2}\overline{\cos^2\theta} - \frac{1}{2}. \quad (14)$$

We now substitute the expansion [9] into the radiative transfer equation (6) and use equation (12) to obtain

$$\nabla\cdot\sum_{l=0}^{\infty}\hat{\omega}\mathbf{P}_l(\hat{\omega})(\cdot)^l\mathbf{i}_l(\mathbf{r}) = ai_b + \sum_{l=0}^{\infty}(s\eta_l - K)\mathbf{i}_l(\mathbf{r})(\cdot)^l\mathbf{P}_l(\hat{\omega}). \quad (15)$$

A simple way to extract from equation (15), equations for the individual coefficients is to take full-sphere moments using $\mathbf{P}_l(\hat{\omega})$ as weight functions, i.e. multiply by $\mathbf{P}_l(\hat{\omega})$ and integrate using orthogonality properties of $\mathbf{P}_l(\hat{\omega})$. This automatically truncates the series. We obtain immediately using $\hat{\omega} = \mathbf{P}_l(\hat{\omega})$ and direct vector integrations, for $l = 0$ and 1

$$\frac{4\pi}{3}\nabla\cdot\mathbf{i}_l \equiv \nabla\cdot\mathbf{q} = 4\pi a(i_b - i_0) \quad (16)$$

$$\nabla i_0 + \frac{3}{5}\nabla\cdot\mathbf{i}_2 = (s\eta_1 - K)\mathbf{i}_1 \equiv \frac{3}{4\pi}(s\delta - K)\mathbf{q}. \quad (17)$$

The procedure of taking the gradient of equation (16) and inserting equation (17) into the result then yields

$$\nabla(\nabla\cdot\mathbf{q}) - 3a(K - s\delta)\mathbf{q} = 4a\nabla e_b + \frac{12\pi a}{5}\nabla\cdot\mathbf{i}_2. \quad (18)$$

This form of the differential 'approximation' for spectral heat flux is evidently valid without approximation for an isotropic medium. Thus, to obtain equation (1) we need only to neglect the term containing $\mathbf{i}_2$, i.e. neglect the shear or non-equilibrium portion of the radiation pressure tensor (and install the suppressed spectral λ subscripts).

2.1. *Relation to one-dimensional heat transfer with scattering*

An alternate approach to expanding the intensity field $i(\mathbf{r}, \hat{\omega})$ in $\mathbf{P}_l(\hat{\omega})$ is to consider direct expansion of the scattering phase function

$$\Phi = 1 + \sum_{l=1}^{\infty}\sum_{n=1}^{\infty} \mathbf{P}_l(\hat{\omega})(\cdot)^l \beta_{ln}(\cdot)^n \mathbf{P}_n(\hat{\omega}_i) \quad (19)$$

where β_{ln} are expansion coefficient tensors. Conditions of isotropy and symmetry require both that $\beta_{ln} = 0$ for $l \neq n$, and that β_{ll} be isotropic symmetric tensors $\beta_{ll} = 2^l(l!)^2\beta_l\delta_{2l}/(2l)!$ with

$$\mathbf{P}_l(\hat{\omega})(\cdot)^l\delta_{2l}(\cdot)^l\mathbf{P}_l(\hat{\omega}_i)$$
$$= \mathbf{P}_l(\hat{\omega})(\cdot)^l\mathbf{P}_l(\hat{\omega}_i) = (2l)!P_l(\hat{\omega}\cdot\hat{\omega}_i)/2^l(l!)^2.$$

Thus, equation (19) becomes for isotropic *media*, the scalar Legendre function expansion, with $\cos\theta = (\hat{\omega}_i\cdot\hat{\omega})$

$$\Phi(\theta) = 1 + \sum_{l=1}^{\infty}\beta_l P_l(\cos\theta). \quad (20)$$

With $\Phi = 4\pi\psi$ and on inserting into equation (13), it follows that the expansions coefficients β_l and the three-dimensional eigenvalues η_l are related by

$$\eta_l = \beta_l/(2l+1). \quad (21)$$

These expansion coefficients, i.e. coefficients of scattering anisotropy, have received attention in direct one-dimensional considerations [19] of the basic radiative transfer equation (6). In this case the intensity field $i(\mathbf{r}, \hat{\omega}) = i(z, \mu)$, with $\mu = \hat{\omega}\cdot\hat{z}$ the direction cosine of $\hat{\omega}$ in the direction of variation $\hat{z}$. Choosing this direction also as the polar axis the scattering integral of equation (6) integrates directly over azimuthal angle, allowing equation (20) to be replaced by the series

$$\Phi = 1 + \beta_1\mu\mu_i + \cdots \quad (22)$$

and the basic radiative transfer equation (6) assumes with this form

$$\mu\frac{\mathrm{d}i}{\mathrm{d}Z} = -Ki + ai_b + \frac{s}{2}\int_{-1}^{1}\Phi i(Z, \mu_i)\,\mathrm{d}\mu_i. \quad (23)$$

Detailed studies of this equation for i in linearly anisotropic scattering, i.e. truncating at the β_l term in equation (23) have been performed by Beach *et al.* [20]. A comparison of equations (3), (7), (14), and (21) shows that their 'anisotropy coefficient' $\beta \equiv \beta_1$ is related to δ_λ as employed in the differential and diffusion approximations—equations (1) and (2) by

$$\delta_\lambda = \beta/3 \quad (24)$$

and β is restricted by equation (3) to the range -3 to 3. The fact that the same truncation is afforded automatically by the above three-dimensional considerations suggests the analysis may be quite accurate for heat transfer analysis with measured values of δ_λ or β. This conclusion is substantiated by the calculations of ref. [13] wherein the coefficient of linear anisotropy, a_1, is interchangeable with β.

2.2. *Boundary conditions*

The non-invariant form of the one-dimensional radiative transfer equation (23) has the well-known advantage over the three-dimensional differential approximation of allowing more flexible boundary conditions for non-optically thick regions of an absorbing, emitting, and scattering medium, wherein the radiant intensity originating from distinct sources can display discontinuities, e.g. hemispherical. However, the use of generalized boundary conditions pioneered by Deissler [21] considerably weakens this advantage. It is therefore of interest to identify any effect of anisotropic scattering within an isotropic medium on these boundary conditions.

We consider an absorbing–emitting–diffusely reflecting boundary through which there is no transmission. Suppressing spectral subscripts and defining outward (q_{+n}) and inward (q_{-n}) portions of the normal heat flux $\mathbf{q}\cdot\hat{n} = q_n$, the boundary balances are

$$q_n = q_{+n} - q_{-n} \quad (25)$$

$$q_{+n} = \sum_{l=0}^{\infty}\mathbf{i}_l(\mathbf{r})(\cdot)^l\int\mathbf{P}_l(\hat{\omega})\hat{\omega}\cdot\hat{n}\,\mathrm{d}\omega \quad \hat{\omega}\cdot\hat{n} > 0 \quad (26)$$

$$q_{+n} = \varepsilon e_{bw} + (1-\varepsilon)q_{-n} \quad (27)$$

where ε is the spectral hemispherical emissivity of the boundary (wall) and e_{bw} is the spectral Planck black body emissive power at that temperature. The $l = 0$ and 1 terms of the complete expansion in (9) and (11) are

$$i(\mathbf{r}, \hat{\omega}) = i_0 + \frac{3}{4\pi}\mathbf{q}\cdot\hat{\omega} + \cdots \quad (28)$$

This substituted into equation (26) yields

$$q_{+n} = \pi i_0 + \frac{1}{2}q_n + \frac{1}{16}\hat{n}\hat{n} : \mathbf{i}_2$$
$$+ \sum_{l=3}^{\infty}\mathbf{i}_l(\mathbf{r})(\cdot)^l\int\mathbf{P}_l(\hat{\omega})\hat{\omega}\cdot\hat{n}\,\mathrm{d}\omega \quad \hat{\omega}\cdot\hat{n} > 0. \quad (29)$$

To obtain closure for a boundary condition we truncate the expansion at the $l = 2$ term, neglect the non-equilibrium radiation-pressure term $\hat{n}\hat{n}:\mathbf{i}_2$, and insert the $l = 0$ moment equation (16). This yields

$$q_{+n} = e_b(\mathbf{r}) - \frac{1}{4a}\nabla\cdot\mathbf{q} + \frac{1}{2}q_n \tag{30}$$

so that on combining with equations (25) and (27) to eliminate q_{+n}, and q_{-n} we recover the standard form on inserting spectral subscripts

$$-\frac{1}{4a_\lambda}\nabla\cdot\mathbf{q}_\lambda + \left(\frac{1}{\varepsilon_\lambda} - \frac{1}{2}\right)\mathbf{q}_\lambda\cdot\hat{n} = e_{bw\lambda} - e_{b\lambda}(\mathbf{r}) \tag{31}$$

in which the right-hand side is the jump condition at the wall in the event conduction is not included. Since the absorption coefficient a_λ appears without a scattering or extinction coefficient, we conclude that neither anisotropic nor isotropic scattering alters the jump boundary condition. This condition is less exact than the differential approximation, equation (1), since the series of equation (29) is not automatically truncated by orthogonality in hemispherical integrations.

3. DISCUSSION OF SOLUTIONS

This section is concerned with the effect of anisotropic scatter coefficient δ in employing the extended differential approximation, equation (1), in several simplistic circumstances in which boundary condition (31) is applied. We confine ourselves here to grey media and hence remove the spectral (λ) designation so that e_b assumes the usual Stefan–Boltzmann form, σT^4.

In the simplest problem of steady state radiative transfer with no sources and radiative equilibrium, $\nabla\cdot\mathbf{q}$ vanishes in equations (1) and (31), hence the problem is equivalent to application of the modified diffusion approximation, equation (2). Thus existing solutions apply subject only to value selection of an equivalent extinction coefficient. However, as soon as distributed heat sources, heat conduction and convection, or thermal transients are incorporated, this simple adjustment is insufficient, and it becomes necessary to develop new solutions making direct modifications at an earlier stage of the development–computation process.

For example, ref. [11] offers two such solutions accounting for linear anisotropic scattering within a one-dimensional plane parallel layer bounded by black walls, where the medium is (i) at radiative equilibrium, and (ii) at constant temperature. The analytical forms for both these solutions as well as the graphical–numerical results are directly applicable to general anisotropic scattering in the differential approximation by substituting δ for their $a_1/3$, cf. equation (24) and subsequent remarks. The solution in case (i) follows directly from the corresponding solution for isotropic scatter by replacing K with $K - s\delta$. In contrast, the case (ii) solution is not obtained this way and represents a new derivation.

Additional solutions with linear anisotropic scattering in one-dimensional cylindrical geometry for three temperature profiles are provided graphically in ref. [13]. These too represent solutions for general anisotropic scattering in the differential approximation by replacing $a_1/3$ with δ.

An interesting example where the effect of scattering can be accounted for simply, concerns the isothermal volumetric melting or freezing of non-opaque materials which is promoted in a two-phase zone by internal radiation. A general description of these types of problems with several one-dimensional applications has been reported by Chan *et al.* [22]. In their applications these authors invoked known radiative transfer solution characteristics of slab geometry involving E functions. Here we present an alternate approach by applying equation (1), primarily for advantage in extending to three-dimensional geometries, including cylindrical and spherical shapes.

3.1. *One-dimensional (slab) geometry*

To make the connection to ref. [22] evident we briefly describe the simplest case, a semiinfinite ice zone at its melting point suddenly subjected to radiation at $Z = 0$ from a grey surface of emissivity ε_w. If α denotes the liquid volume fraction, λ the latent heat of melting, ρ the density of liquid and solid, then the transient heat balance, equations (1) and (31), reduces respectively to

$$\rho\lambda\frac{\partial\alpha}{\partial t} = -\frac{\partial q}{\partial Z} \tag{32}$$

$$\frac{\partial^2 q}{\partial Z^2} = b^2 q \tag{33}$$

$$-\frac{1}{4a}\left(\frac{\partial q}{\partial Z}\right)_0 + \left(\frac{1}{\varepsilon_w} - \frac{1}{2}\right)q = \sigma(T_w^4 - T_m^4) \tag{34}$$

with

$$b^2 = 3a(K - s\delta). \tag{35}$$

In general b^2 varies with α due to variations of a, s, and δ. As the simplest approximation we treat it as constant at an intermediate mean value. Then the solution assumes the exponential form for time $t < t_0$

$$\alpha = (t/t_0)\,e^{-bZ} \tag{36}$$

$$q = (\rho\lambda/bt_0)\,e^{-bZ}. \tag{37}$$

Here

$$t_0 = \frac{\rho\lambda}{b\sigma(T_w^4 - T_m^4)}\left(\frac{b}{4a} + \frac{1}{\varepsilon_w} - \frac{1}{2}\right) \tag{38}$$

is the time delay required to approach complete melting at the $Z = 0$ boundary and thereby establish a full wave exponential melt profile e^{-bZ}. Our t/t_0 compares directly with t^* of ref. [22], as may be seen by neglecting scatter so that $b = 3a$. For times $t > t_0$ the melt propagates into the medium. If the fully melted layer which forms behind the profile is assumed either to be continuously removed or alternatively to transmit radiation through it to the advancing profile, then the established exponential shape holds during propagation, and the flux profile (37) remains as

$$q = (\rho\lambda/bt_0)\,e^{-b(Z-Z_0)} \tag{39}$$

with $Z_0(t)$ the junction between the fully melted layer and the wave beginning. It follows from equation (32) that within the wave ($Z > Z_0$)

$$\alpha = e^{-bZ} + e^{-bZ}\int_{t_0}^{t} e^{bZ_0(t')}\,dt'/t_0$$

from which we find using $\alpha = 1$ at $Z = Z_0$ that $Z_0(t)$ propagates at a constant rate defined by

$$Z_0(t) = (t-t_0)/bt_0 \quad t \geqslant t_0 \tag{40}$$

and hence the melt profile propagation is described for $Z > Z_0$ by

$$\alpha = e^{-b(Z-Z_0(t))}. \tag{41}$$

The total amount of melt contained in the advancing layer is evidently $1/b$ as determined by scatter and absorption through use of equation (35).

3.2. *Volumetric thawing of an ice rod in internal radial irradiation*

This section extends the melting analysis of the previous section to inward cylindrical melting. The rod is initially at its melting temperature, T_m. Here the analogues of equations (32)–(34) in terms of $X = br$ with r the radial coordinate, are

$$\frac{\partial\alpha}{\partial t} = -\frac{b}{\rho\lambda X}(Xq)' \tag{42}$$

$$(X^{-1}(Xq)')' = q \tag{43}$$

$$-\frac{b}{4aX}(Xq)' - \left(\frac{1}{\varepsilon_w} - \frac{1}{2}\right)q = \sigma(T_w^4 - T_m^4). \tag{44}$$

Here primes denote differentiation, d/dX. Equation (44) is applied at $X = X_R = bR$, with R the initial rod radius, until complete melting ($\alpha = 1$) is reached there at time t_0. During the melt zone formation period ($t \leqslant t_0$) equations (43) and (42) integrate in this order to

$$q = -\frac{\rho\lambda}{b}\psi(X_R)I_1(X) \tag{45}$$

$$\alpha = (t/t_0)I_0(X)/I_0(X_R) \tag{46}$$

with I_0, I_1 modified Bessel functions of order 0, 1 of the first kind. (Note that radial radiation flux q is negative, being defined in the positive r-direction.) The coefficient $\psi(X_R)$ and time t_0 are determined respectively as

$$\psi(X) = \frac{b}{\rho\lambda}\sigma(T_w^4 - T_m^4)\left[\frac{b}{4a}I_0(X) + \left(\frac{1}{\varepsilon_w} - \frac{1}{2}\right)I_1(X)\right]^{-1} \tag{47}$$

and

$$t_0 = \frac{1}{\psi(X_R)I_0(X_R)}. \tag{48}$$

At the center line ($X = 0$) the melt fraction for the formation period $t \leqslant t_0$ follows immediately by equation (46) using $I_0(0) = 1$.

For times $t \geqslant t_0$ we suppose that the fully melted layer is either sufficiently drained or transparent such that the boundary condition, equation (44), continues to hold at the outer edge, $X = X_0(t)$, of the inward moving melt wave. Equation (45) then immediately applies as

$$q = \frac{\rho\lambda}{b}\psi(X_0)I_1(X) \tag{49}$$

so that for $X < X_0$, $\partial\alpha/\partial t = \psi(X_0)I_0(X)$; hence

$$\alpha = \frac{I_0(X)}{I_0(X_R)} + I_0(X)\int_{t_0}^{t}\psi(X_0(t'))\,dt'. \tag{50}$$

Into this result, we insert the condition $\alpha = 1$ at $X = X_0$, divide by $I_0(X_0)$, differentiate in time t, separate and integrate using $(1/I_0(X))' = I_1(X)/I_0^2(X)$, to obtain

$$t - t_0 = \int_{X_0}^{X_R} dX\frac{I_1(X)}{I_0^2(X)\psi(X)}. \tag{51}$$

This defines $t(X_0)$ or $X_0(t)$ as a quadrature integral, using definition (47). Such integration also determines the time for complete melting by choosing $X_0 = 0$. Similarly using $\psi(X_0(t'))\,dt' = (1/I_0(X_0))'\,dX_0$, the melt profile from equation (50) reduces to

$$\alpha = I_0(X)/I_0(X_0) \tag{52}$$

applicable for $t > t_0$, $X < X_0(t)$. The center line melt level for $t > t_0$ is therefore just $1/I_0(X_0)$ as determined by equation (51).

3.3. *Solidification in a spherical cloud of molten droplets*

For this example, we consider the droplets to be initially all liquid at their liquidus or freezing temperature T_m, and to be uniformly and homogeneously dispersed within a transparent medium with cloud radius R. Their size distribution is such that only a negligible mass fraction of them are so small that conduction–convection transfers significant latent or sensible heat locally to the medium in comparison to the bulk heat removal by internal cloud radiation flux q. The medium itself is therefore nonparticipating and may be at substantially lower temperature.

Under these idealizations the analysis becomes an extension to spherical symmetry of the above two examples, with $\rho = \rho_p\alpha'$, where ρ_p is both droplet and frozen particle density, and α' is the droplet volume fraction within the cloud; α becomes the average fraction of a droplet in the liquid state at radial position r, so that $\alpha = 1$ initially for $0 < r < R$. In terms of $X = bR$, $X_R = bR$, cf. equation (35), the equations to solve analogous to equations (42)–(44) become

$$\frac{\partial \alpha}{\partial t} = \frac{-b}{\rho \lambda X^2}(X^2 q)' \quad (53)$$

$$(X^{-2}(X^2 q)')' = q \quad (54)$$

$$\frac{b}{4aX^2}(X^2 q)' + \left(\frac{1}{\varepsilon_w} - \frac{1}{2}\right) q = \sigma(T_m^4 - T_w^4) \quad (55)$$

where equation (55) is applied at $X = X_R$ until α reaches zero there (at time t_0). For time $t < t_0$, we thereby obtain in terms of the hyperbolic functions

$$q = \frac{\rho\lambda}{b}\psi(X_R)(X\cosh X - \sinh X)/X^2 \quad (56)$$

$$\frac{\partial \alpha}{\partial t} = -\psi(X_R)\sinh X/X. \quad (57)$$

Here, for $\varepsilon_w = 1$

$$\psi(X) = \frac{2bX^2\sigma}{\rho\lambda}(T_m^4 - T_w^4)\left[X\cosh X + \left(\frac{bX}{2a} - 1\right)\sinh X\right]^{-1} \quad (58)$$

so that the freeze fraction profile for $t \leqslant t_0$, $X \leqslant X_R$, is

$$\alpha = 1 - \frac{t}{t_0}\frac{X_R}{X}\frac{\sinh X}{\sinh X_R} \quad (59)$$

with

$$t_0 = \frac{X_R}{\psi(X_R)\sinh X_R}. \quad (60)$$

Equation (59) immediately yields the cloud center point freeze fraction using $\sinh X/X = 1$ at $X = 0$. In particular, this becomes

$$\alpha = 1 - X_R/\sinh X_R \quad (61)$$

at time t_0 where freezing is just completed at the cloud boundary X_R. After this time it is necessary to account for an outer zone of falling frozen droplet temperatures; hence we do not extend the isothermal wave propagation analysis of the above examples to present circumstances. The effects of scattering as contained in the coefficient b, cf. equation (35), for these types of problems will be much less sensitive to variations in α in more exact analysis and computations. An ultrasimplified example treating spherically symmetric temperature variations is illustrated below.

3.4. *Steady state emitting sphere with volumetric heat source*

As a final simple example we consider spherically symmetric internal radiation with temperature variations controlled by a constant volumetric heat source S. This could be an idealized cloud of droplets or particles dispersed in transparent gases, or else a more homogeneous non-opaque medium, provided only that the temperature is not internally defined, e.g. by a melt–freezing phase change.

In the absence of conduction and convection, with constant S (radius $r \leqslant R$) the solution to this problem for the temperature field and radial heat flux is simply

$$q_R = \frac{SR}{3} = \sigma(T_R^4 - T_w^4)\left(\frac{3}{4Ra} + \frac{1}{2}\right)^{-1} \quad (62)$$

$$T_0^4 - T^4 = \frac{Sr^2}{8\sigma}(K - s\delta). \quad (63)$$

The outer boundary temperature T_R (and, of course, heat flux q_R) at steady state is unaffected by scattering, either isotropic or anisotropic. However, the temperature profile from $r = R$ inward is altered as shown by equation (63). The relation between center point temperature predicted with and without anisotropic scattering is evidently

$$\frac{T_{\text{osca}}^4 - T_R^4}{T_0^4 - T_R^4} = 1 - \Omega\delta \quad (64)$$

corresponding to higher center point temperatures for backward scatter ($\delta < 0$) and lower values for forward scatter as expected. The scattering albedo $\Omega = s/K$ determines the importance of the effect. This simple result also follows from the diffusion approximation, equation (2), since $\nabla(\nabla \cdot \mathbf{q})$ vanishes in equation (1). Such equivalence does not carry over, however, as soon as variations of source S or transients are considered.

4. CONCLUDING REMARKS

In Section 1 the invariance of the pressure cross-section to the inclusion of forward scatter, a characteristic not shared by total extinction cross-section, was observed as an additional justification for recognizing the pressure cross-section as the proper one to use in the differential approximation for heat transfer, and also as fundamental to use of the Mie scattering solution to define radiative transport coefficients. The basis for this invariance is explained by observing a corresponding fundamental invariance property of the radiative transport equation (6). If a component of forward scatter is added to the phase function $\psi(\cos\theta)$, then to assure normalization the new phase function must be of the form

$$\psi' = c\psi + (1-c)\delta(\cos\theta - 1)/2\pi. \quad (65)$$

When this expression is inserted into the scattering terms of equation (6), namely

$$s\left[\int \psi(\cos\theta) i(\hat{\omega})\,d\hat{\omega} - i(\hat{\omega}_i)\right]$$

it follows that this expression retains its value if the scattering coefficient, and total extinction coefficient are changed to new values defined by

$$s' = s/c \quad (66)$$

$$K' = a + s/c = a + (K-a)/c. \quad (67)$$

By transforming the scattering and extinction coefficients in this manner, it is readily verified that $s(1-\overline{\cos\theta})$, and hence $K-s\delta$ is invariant to the transformation of equation (65).

The preceding demonstrates how the powerful 'differential approximation' for radiation heat transfer in an absorbing, emitting, and scattering medium may readily be formulated and applied to properly account for anisotropic scattering in three-dimensional problems with or without conditions of large optical depths.

The key to the simple elegant derivation of this is the use of the coordinate-system invariant, i.e. rotationally invariant, direct vector–tensor formulation. Polyadic Legendre functions provide, in this approach, an alternative to the use of coordinate-system specific 'P_N approximations' and spherical harmonics expansions. The orthogonality properties of these polyadic functions establish the general nature of the differential approximation for anisotropic scattering as free from arbitrary truncations. When this is recognized equation (1) may be reproduced using a very simple truncated moment scheme with direct-vector methods. We anticipate that invariant formulations can be used to advantage in the development of solutions to the field equation (6) for spectral intensity when proceeding beyond the extensively studied one-dimensional slab geometry.

While this paper is limited for simplicity to isotropic media, scattering sites themselves, e.g. particles of a particle cloud, need not be rotationally invariant or spheres. They can also be orientable, e.g. rods, spherocylinder capsules, etc., provided their orientation is distributed randomly. For non-random distributions, the medium is anisotropic. More complex differential approximations for anisotropic media may be derived from a correspondingly generalized field equation (6) to account for the effect of preferred particle orientations on absorption and scattering. Such generalized studies will complement and provide natural coupling of radiation heat transfer to previous studies of other transport modes in systems of orientable particles [23–25].

Acknowledgment—Appreciation is expressed to Professor S. H. Chan for interesting and useful comments and Dr D. H. Cho for encouragement.

REFERENCES

1. M. P. Mengüc and R. Viskanta, Radiative transfer in three dimensional rectangular enclosures containing inhomogeneous, anisotropic scattering media, *J. Quantve Radiat. Transfer* **33**, 533 (1985).
2. A. C. Ratzel, III and J. R. Howell, Two dimensional energy transfer in radiatively participating media with conduction by a *P–N* approximation. In *Heat Transfer 82*, Vol. 2, p. 535. Hemisphere, Washington, D.C. (1982).
3. M. M. Razzaque, D. E. Kline and J. R. Howell, Finite element solution of radiative heat transfer in a two-dimensional rectangular enclosure with grey participating media, *J. Heat Transfer* **105**, 933 (1982).
4. M. M. Razzaque, J. R. Howell and D. E. Kline, Finite element solution of combined radiative convection and conduction heat transfer, *Trans. Am. Nucl. Soc.* **38**, 334 (1981).
5. R. L. Fernandes, J. Francis and J. N. Reddy, A finite-element approach to combined conduction and radiative heat transfer in a planar medium. In *Progress in Astronautics and Aeronautics* (Edited by A. L. Crosbie), Vol. 78, pp. 92–109. AIAA, New York (1980).
6. R. Fernandes and J. Francis, Combined conductive and radiative heat transfer in an absorbing, emitting and scattering cylindrical medium, *J. Heat Transfer* **104**, 594 (1982).
7. S. T. Wu, R. E. Ferguson and L. L. Altgilber, Application of finite element techniques to the interaction of conduction and radiation in participating media. In *Progress in Astronautics and Aeronautics* (Edited by A. L. Crosbie), Vol. 78, pp. 61–91. AIAA, New York (1980).
8. W. A. Fiveland, Discrete ordinates solutions of the radiative transport equation for rectangular coordinates, *J. Heat Transfer* **106**, 699 (1984).
9. R. Siegel and J. R. Howell, *Thermal Radiation Heat Transfer*, 2nd edn. McGraw-Hill, New York (1982).
10. E. M. Sparrow and R. D. Cess, *Radiation Heat Transfer*. Brooks/Cole, Belmont, California (1970).
11. M. F. Modest and F. H. Azad, The differential approximation for radiative transfer in an emitting, absorbing and anisotropically scattering medium, *J. Quantve Spectrosc. Radiat. Transfer* **23**, 117 (1980).
12. M. F. Modest and F. H. Azad, The influence and treatment of Mie–anisotropic scattering in radiative heat transfer, *J. Heat Transfer* **102**, 92 (1980).
13. F. H. Azad and M. F. Modest, Evaluation of the radiative heat flux in absorbing, emitting and linear-anisotropically scattering cylindrical media, *J. Heat Transfer* **103**, 350 (1981).
14. F. H. Azad, Differential approximation to radiative transfer in semi-transparent media, *J. Heat Transfer* **107**, 478 (1985).
15. H. C. Van de Hulst, *Light Scattering by Small Particles*. Wiley, New York (1957).
16. H. C. Hottel and A. F. Sarofim, *Radiative Transfer*. McGraw-Hill, New York (1967).
17. H. Brenner, The Stokes resistance of a slightly deformed sphere—II. Intrinsic resistance operators for an arbitrary initial flow, *Chem. Engng Sci.* **22**, 375 (1967).
18. W. D. Henline and D. W. Condiff, Transport mechanics in systems of orientable particles. II. Kinetic theory of orientation specific transport for hard-core models, *J. Chem. Phys.* **52**, 5027 (1970).
19. M. N. Özisik, *Radiative Transfer*. Wiley, New York (1973).
20. H. L. Beach, M. N. Özisik and C. E. Siewert, Radiative transfer in linearly anisotropic scattering, conservative and non-conservative slabs with reflective boundaries, *Int. J. Heat Mass Transfer* **14**, 1551 (1971).
21. R. G. Deissler, Diffusion approximation for thermal radiation in gases with jump boundary condition, *J. Heat Transfer* **86**(2), 240 (1964).
22. S. H. Chan, D. H. Cho and G. Kocamustafaogullari, Melting and solidification with internal radiative transfer—a generalized phase change model, *Int. J. Heat Mass Transfer* **26**, 621 (1983).
23. D. W. Condiff and H. Brenner, Transport mechanics in systems of orientable particles, *Physics Fluids* **12**(3), 539 (1969).
24. H. Brenner and D. W. Condiff, Transport mechanics in systems of orientable particles III. Arbitrary particles, *J. Coll. Interfacial Sci.* **41**(2), 228 (1972).
25. H. Brenner and D. W. Condiff, Transport mechanics in systems of orientable particles IV. Convective transport, *J. Coll. Interfacial Sci.* **47**(1), 199 (1974).

DIFFUSION ANISOTROPE DANS L'APPROXIMATION TRIDIMENSIONNELLE DU RAYONNEMENT THERMIQUE

Résumé—L'approximation différentielle est étendue à la diffusion anisotrope dans une forme invariante tridimensionnelle. Une méthode de moment utilisant les fonctions polyadiques de Legendre établit que les sections droites de pression doivent avoir la préséance vis-à-vis des sections droites d'extinction pour traiter le rayonnement thermique dans un milieu absorbant, émetteur et diffusif et cette utilisation des sections droites rend compte de la diffusion avant ou arrière. On discute de la généralisation de l'approximation dans quelques cas simples.

EINE DREIDIMENSIONALE DIFFERENTIELLE APPROXIMATION DER ANISOTROPEN STREUUNG DER WÄRMESTRAHLUNG

Zusammenfassung—Die differentielle Approximation wird für anisotrope Streuung in invarianter dreidimensionaler Form erweitert. Eine Momenten-Methode unter Verwendung von polyadischen Legendre-Funktionen zeigt, daß bei der Behandlung des Wärmeaustausches durch Strahlung in absorbierenden, emittierenden und streuenden Medien die Druckquerschnitte den Extinktionsquerschnitten vorgezogen werden sollten. Der Gebrauch dieser Querschnitte berücksichtigt die Bevorzugung von Vorwärts- oder Rückwärts-Streuung. Die ermittelte verallgemeinerte differentielle Approximation wird anhand mehrerer einfacher Bespiele diskutiert.

ПРОСТРАНСТВЕННАЯ ДИФФЕРЕНЦИАЛЬНАЯ АППРОКСИМАЦИЯ АНИЗОТРОПНОГО РАССЕЯНИЯ ПРИ РАДИАЦИОННОМ ТЕПЛОПЕРЕНОСЕ

Аннотация—Дифференциальные аппроксимации применяются для учета анизотропного рассеяния в инвариантной трехмерной форме. С помощью метода моментов с использованием полиномов Лежандра установлено, что сечения, в которых происходит усиление, должны преобладать над сечениями, в которых наблюдается ослабление при радиационном переносе тепла в поглощающих, излучающих и рассеивающих средах. Такой расчет поперечных сечений используется для установления степени превалирования рассеяния вперед или назад. Обобщенные дифференциальные аппроксимации обсуждаются на примерах различных простых задач.